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A Contour Integral Representation of Euler–Frobenius Polynomials

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1. INTRODUCTION

The Euler–Frobenius polynomials $(p_m)_{m\geqslant 1}$ have been introduced by Euler in 1755 and thoroughly investigated by Frobenius [2]. Later on it was Schoenberg [6–8] who pointed out the important rôle that these remarkable polynomials play in the theory of cardinal spline functions. In particular, Schoenberg [7, 8] has proved that for any complex weight $h \neq 1$ that is different from all the zeros of p_m there exists one and only one cardinal exponential spline interpolant s_m on $\mathbb{R}$ of degree $m \geqslant 1$ with respect to the bi-infinite geometric sequence $(h^n)_{n\in\mathbb{Z}}$.

A previous paper [4] has been concerned with a contour integral representation of cardinal exponential splines (cf. Theorem 1). Working in the same vein as in [4], it is the purpose of the present paper to establish a contour integral representation of the Euler–Frobenius polynomials $(p_m)_{m\geqslant 1}$ (Theorem 3) and to deduce from it their main properties by the methods of complex analysis. Thus, the present paper may be considered as a supplement to [4] which aims to illustrate the effectiveness of the contour integral representation approach to some problems arising in the theory of cardinal spline functions.

2. CARDINAL EXPONENTIAL SPLINE FUNCTIONS

Let $m \geqslant 1$ be a fixed integer and let $\mathfrak{S}_m(\mathbb{R}; \mathbb{Z})$ denote the complex vector space of all cardinal spline functions of degree m on $\mathbb{R}$ with respect to the grid $\mathbb{Z}$ as their knot sequence. It is well known (Curry and Schoenberg [1], Schoenberg [7, 8]) that there exists a unique spline function $b_m \in \mathfrak{S}_m(\mathbb{R}; \mathbb{Z})$ such that $\operatorname{Supp}(b_m) = [\![0, m+1]\!]$, $\int_{\mathbb{R}} b_m(t)\,dt = 1$ and its translates $\{b_m(\cdot - n) \mid n \in \mathbb{Z}\}$ form a basis of the vector space $\mathfrak{S}_m(\mathbb{R}; \mathbb{Z})$ over $\mathbb{C}$.

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Let the complex number $h \neq 0$ be fixed. A distinguished one-dimensional vector subspace of $\mathfrak{S}_m(\mathbb{R}; \mathbb{Z})$ is spanned by the spline $\sum_{n \in \mathbb{Z}} h^n b_m(\cdot - n)$. The elements of this vector subspace are called *cardinal exponential splines* of degree m and weight h (Schoenberg [7, 8]). In order that the element

$$s_m = C_m \sum_{n \in \mathbb{Z}} h^n b_m(\cdot - n) \tag{1}$$

should be a cardinal exponential spline interpolant of degree m with respect to the bi-infinite geometric sequence $(h^n)_{n \in \mathbb{Z}}$ a necessary and sufficient condition is that the constant $C_m \in \mathbb{C}^\times$ may be determined so that

$$s_m(0) = 1 \tag{2}$$

holds.

In the case when the weight $h \neq 0$ does not belong to the unit circle $U = \{z \in \mathbb{C} \mid |z| = 1\}$, a contour integral representation of the cardinal exponential splines $s_m \in \mathfrak{S}_m(\mathbb{R}; \mathbb{Z})$ of degree $m \geqslant 1$ is possible. In the case when $|h| > 1$ choose two real numbers c, c' so that $0 < c < \log |h| < c'$ holds. In the other case when $0 < |h| < 1$ suppose that the real numbers c, c' satisfy the conditions $c < \log|h| < c' < 0$. In any case, introduce the two vertical lines

$$L = \{z \in \mathbb{C} \mid \operatorname{Re} z = c\}, \qquad L' = \{z \in \mathbb{C} \mid \operatorname{Re} z = c'\}$$

in the complex plane $\mathbb{C}$. Then $L \cup L'$ forms the boundary of a closed vertical strip in the open complex right, resp. left, half-plane with the compact basis $[\![c, c']\!]$ on the real axis $\mathbb{R}$. Let the lines L, L' be equipped with an orientation so that their juxtaposition

$$P = L \vee L' \tag{3}$$

forms a cycle in the extended complex plane that admits the topological index

$$\operatorname{Ind}_P(\log |h|) = 1 \tag{4}$$

with respect to the point $\log |h| \in [\![c, c']\!]$ on $\mathbb{R}$. Then we may state

THEOREM 1. *Any cardinal exponential spline $s_m \in \mathfrak{S}_m(\mathbb{R}; \mathbb{Z})$ of degree $m \geqslant 1$ and weight $h \in \mathbb{C}^\times - U$ admits the contour integral representation with transcendental meromorphic integrand*

$$s_m : \mathbb{R} \ni x \rightsquigarrow C_m \left(1 - \frac{1}{h}\right)^{m+1} \frac{1}{2\pi i} \int_P \frac{e^{(x+1)z}}{(e^z - h) z^{m+1}} \, dz, \tag{5}$$

where $C_m \in \mathbb{C}$ denotes an arbitrary constant. The contour integral occurring in (5) is independent of the particular choices of c and c'.

We shall not stop to repeat the proof. It is based on certain line integral representations of the basis spline b_m that are obtained by means of the inverse Laplace transform. Details may be found in [4] or [5].

Observe that the meromorphic function

$$z \rightsquigarrow \frac{e^{(x+1)z}}{(e^z - h)z^{m+1}} \qquad (x \in \mathbb{R}) \tag{6}$$

has a pole of order $m+1$ at the origin of $\mathbb{C}$ and simple poles at the zeros of the function $z \rightsquigarrow e^z - h$ with period $2\pi i$. In the case $x \in [\![-1, 0]\!]$ the sum of all the residues of (6) in $\mathbb{C}$ vanishes (Schoenberg [7]). Thus, according to the residue theorem and the Cauchy integral formulae the contour integral

$$-\frac{1}{2\pi i}\int_P \frac{e^{(x+1)z}}{(e^z - h)z^{m+1}}\,dz \qquad (m \geqslant 1)$$

along the path (3) with the orientation defined by (4) represents the mth coefficient in the power series expansion of the meromorphic function

$$z \rightsquigarrow \frac{e^{(x+1)z}}{e^z - h} \qquad (x \in [\![-1, 0]\!])$$

in a neighborhood of the origin. According to (5) the uniqueness of the local power series expansions implies the following result:

THEOREM 2. *Let $h \in \mathbb{C}^\times - U$ and $x \in [\![-1, 0]\!]$ be given. For all $z \in \mathbb{C}$ so that $|z| < |\log |h||$ the cardinal exponential splines $s_m \neq 0$ $(m \geqslant 1)$ admit the power series expansions*

$$\frac{e^{(x+1)z}}{h - e^z} = \sum_{m \geqslant 0} \frac{h^{m+1}}{C_m(h-1)^{m+1}}\, s_m(x)\, z^m, \tag{7}$$

where $C_0 = 1$ and $s_0 = 1/h$.

The power series expansions (7) in the special case $x = 0$ will be of particular importance in connection with the Euler–Frobenius polynomials defined in the next section; see formula (13) infra.

3. Euler–Frobenius Polynomials

For any integer $m \geqslant 1$ the mth *Euler–Frobenius polynomial* $p_m \in \mathbb{Z}[h]$ is defined according to

$$p_m(h) = m! \sum_{0 \leqslant n \leqslant m-1} b_m(n+1)\, h^n. \tag{8}$$

Observe that p_m is a monic polynomial of degree $m-1$ with strictly positive integer coefficients that satisfies $p_m(0) = 1$ $(m \geqslant 1)$. Moreover, we have

THEOREM 3. *For any $h \in \mathbb{C}^\times - U$ the Euler–Frobenius polynomial p_m of degree $m-1$ $(m \geqslant 1)$ admits the contour integral representation*

$$p_m(h) = \frac{(h-1)^{m+1}}{h} \frac{m!}{2\pi i} \int_P \frac{e^z}{(e^z - h) z^{m+1}}\, dz \qquad (m \geqslant 1), \tag{9}$$

where P is the boundary (3) *of a closed vertical strip in the open complex right, resp. left, half-plane according to the cases $|h| > 1$, resp. $0 < |h| < 1$, equipped with the orientation so that* (4) *holds.*

Proof. If (1) and (5) are evaluated at the point $x = 0$ we obtain the equality

$$\sum_{1 \leqslant n \leqslant m} b_m(n)\, h^{-n} = \left(1 - \frac{1}{h}\right)^{m+1} \frac{1}{2\pi i} \int_P \frac{e^z}{(e^z - h) z^{m+1}}\, dz \tag{10}$$

for $m \geqslant 1$. Since the basis spline b_m satisfies the homogeneous linear difference equation

$$b_m(m+1-x) - b_m(x) = 0 \qquad (x \in \mathbb{R}) \tag{11}$$

we conclude from (8) and (10) that (9) holds. ∎

The symmetry between the two cases $|h| > 1$ and $0 < |h| < 1$ may be expressed by the following reflection principle:

COROLLARY 1. *For all $h \in \mathbb{C}^\times$ the Frobenius reciprocal identity*

$$h^{m-1} p_m\left(\frac{1}{h}\right) = p_m(h) \qquad (m \geqslant 1) \tag{12}$$

holds.

Of course, the identity (12) is nothing more than the homogeneous linear difference equation (11) in terms of the Euler–Frobenius polynomials $(p_m)_{m\geqslant 1}$.

A local power series argument similar to that in the proof of Theorem 2 leads to

COROLLARY 2. *For any $h \in \mathbb{C}^\times - U$, the Euler–Frobenius polynomials $(p_m)_{m\geqslant 1}$ are generated by the power series expansion*

$$\frac{(h-1)e^z}{h(h-e^z)} = \sum_{m\geqslant 0} \frac{p_m(h)}{(h-1)^m} \frac{z^m}{m!} \qquad (|z| < |\log |h||), \tag{13}$$

where $p_0(h) = 1/h$.

In the case when $z \in \mathbb{C}$ satisfies $|z| < |\log |h||$, the power series expansion of the left-hand side of (13) about the origin yields the identities

$$(h-1) \sum_{\nu\geqslant 0} (\nu+1)^m h^{-2-\nu} = \frac{p_m(h)}{(h-1)^m} \qquad (m \geqslant 0).$$

An application of (12) proves

COROLLARY 3. *For any $h \neq 1$, the Euler–Frobenius polynomials $(p_m)_{m\geqslant 1}$ admit the formal power series expansions*

$$\frac{p_m(h)}{(1-h)^{m+1}} = \sum_{n\geqslant 0} (n+1)^m h^n \qquad (m \geqslant 1). \tag{14}$$

4. THE MAIN PROPERTIES OF THE EULER–FROBENIUS POLYNOMIALS

If we compare, for instance, the contour integral representations (5) and (9) of the cardinal exponential splines and the Euler–Frobenius polynomials, respectively, the condition (2) entails

THEOREM 4. *For any $h \in \mathbb{C}^\times - U$ there exists a cardinal exponential spline interpolant of degree $m \geqslant 1$ with respect to the bi-infinite geometric sequence $(h^n)_{n\in\mathbb{Z}}$ if and only if the condition*

$$p_m(h) \neq 0 \tag{15}$$

is satisfied.

The identity (14) furnishes

$$-h(h-1)p'_m(h) = (1-h)^{m+2} \sum_{n \geqslant 0} n(n+1)^m h^n - (m+1)(1-h)^{m+1} \times \sum_{n \geqslant 0} (n+1)^m h^{n+1}$$

$$= p_{m+1}(h) - p_m(h) - mhp_m(h)$$

for $m \geqslant 1$. This proves

THEOREM 5. *The Euler–Frobenius polynomials $(p_m)_{m \geqslant 1}$ satisfy the three-terms recurrence relation*

$$p_{m+1}(h) = (mh+1)p_m(h) - h(h-1)p'_m(h) \qquad (m \geqslant 1). \tag{16}$$

Finally, Theorem 5 implies as an easy consequence

THEOREM 6. *For every integer $m \geqslant 2$ all the roots of the Euler–Frobenius polynomial p_m are simple and located on the open negative real half-line $\mathbb{R}_-^\times$. If $h_0 \in \mathbb{R}_-^\times$ is a root of p_m then its reciprocal $1/h_0 \in \mathbb{R}_-^\times$ is also a root of p_m and $p_m(-1) = 0$ holds if and only if m is even.*

Proof. Since we have $p_2(h) = h + 1$, the statement in the case $m = 2$ is trivial. Suppose that the assertion is verified for an arbitrary integer $m \geqslant 2$. The recurrence relation (16) then shows that p_{m+1} admits alternating signs at the $m-1$ different roots of p_m on $\mathbb{R}_-^\times$. By the intermediate value theorem, the polynomial p_{m+1} of degree m has exactly m different roots on $\mathbb{R}_-^\times$ that separate the $m-1$ roots of p_m. An application of the Frobenius reciprocal identity (12) completes the proof. ∎

In view of the condition (15), the weights $h \in \mathbb{C}^\times - U$ that belong to the complex plane cut along the closed negative real half-line $\mathbb{R}_-$ are admissible choices to construct cardinal exponential spline interpolants with respect to the bilateral geometric sequence $(h^n)_{n \in \mathbb{Z}}$.

For a survey of the contour integral representation approach to the theory of cardinal spline functions, the reader is referred to [5]. In this connection also see [3] where a complex contour integral representation of the cardinal logarithmic splines is established.

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